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Nonlocal elasticity and related variational principles

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Abstract

The Eringen model of nonlocal elasticity is considered and its implications in solid mechanics studied. The model is refined by assuming an attenuation function depending on the 'geodetical distance' between material particles, such that in the diffusion processes of the nonlocality effects certain obstacles as holes or cracks existing in the domain can be circumvented. A suitable thermodynamic framework with nonlocality is also envisaged as a firm basis of the model. The nonlocal elasticity boundary-value problem for infinitesimal displacements and quasi-static loads is addressed and the conditions for the solution uniqueness are established. Three variational principles, nonlocal counterparts of classical ones, i.e. the total potential energy, the complementary energy, and the mixed Hu–Washizu principles, are provided. The former is used to derive a nonlocal-type FEM (NL-FEM) in which the (symmetric) global stiffness matrix reflects the nonlocality features of the problem. An alternative standard-FEM-based solution method is also provided, which consists in an iterative procedure of the type local prediction/nonlocal correction, in which the nonlocality is simulated by an imposed-like correction strain. The potentialities of these analysis methods are pointed out, their numerical implementations being the object of an ongoing research work.

Keywords: Nonlocal elasticity; Variational principles; FEM

1. Introduction

The nonlocal elasticity theory can be traced back to Kröner (1967), who formulated a continuum theory for elastic materials with long range cohesive forces. The latter theory is closely related to the multipolar elasticity theory of Green and Rivlin (1965), as long with the continuum theory derived by Krumhansl (1968) from lattice theory. Eringen (1972a,b, 1976), Eringen and Edelen (1972) and Edelen et al. (1971) gave nonlocal elasticity theories characterized by the presence of nonlocality residuals of fields (like body force, mass, entropy, internal energy,...) and determined these residuals, along with the constitutive laws, by means of suitable thermodynamic restrictions. Eringen and co-workers (see e.g. Eringen and Kim, 1974; Eringen et al., 1977; Eringen, 1978, 1979) simplified the above mentioned theory for linear homogeneous isotropic nonlocal elastic solids, in such a way that the nonlocal theory differ from the classical one in the

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stress–strain constitutive relations only, with the elastic moduli being some simple functions of the Euclidean distance between the strain and the stress points. That enabled these authors and others (see e.g. Artan and Yelkenci, 1996; Zhou et al., 1999) to address many problems that lead to stress singularities in local elasticity (as, typically, the crack-tip problem), and to show that these singularities disappear with the nonlocal treatment. Gao (1999) provided a general theory of nonlocal elasticity which, similarly to a theory mentioned above (e.g. Eringen, 1976), incorporates a set of nonlocality residuals of fields, including the nonlocal couple stress consequent to the effect of local rotation; when the latter effect is neglected, the Eringen (1976) theory is recovered.

All the above referenced works are mainly devoted to the constitutive aspects of the nonlocal elasticity model and to its physical bases. Other aspects of the theory, related to continuum boundary-value and initial-boundary-value problems, were also addressed. Rogula (1982) developed a nonlocal theory of elastic continua and investigated the conditions for the existence of fundamental solutions; Altan (1989a,b) studied questions as the existence and uniqueness of solution; Wang and Dhaliwal (1993a) established a reciprocity relation for nonlocal polar continua. Similar issues were also addressed for nonlocal thermoelasticity by Altan (1991) and Wang and Dhaliwal (1993b).

In consideration of the importance of nonlocal elasticity in solving elasticity problems that lead to stress singularities in the realm of local elasticity, as typically fracture mechanics problems, there is enough motivation for research work on the afore-mentioned theoretical aspects of nonlocal elasticity. Since in the nonlocal approach to elasticity problems with the presence of sharp geometrical singularities or cracks no stress singularities arise, stress-based failure criteria would be meaningful (Eringen et al., 1977; Eringen, 1978) and applicable to fracture and plastic yielding as well.

The nonlocal elasticity considered previously can be qualified as ‘integral’ (or ‘strongly nonlocal’) because it expresses the stress at a point of a material domain as a weighted value of the entire strain field. Another model of nonlocal elasticity, called ‘gradient’ (or ‘weakly nonlocal’), expresses the stress as a function of the strain and its gradients at the same point, see e.g. Kröner (1967) and Rogula (1982). Aifantis and co-workers (see e.g. Vardulakis et al., 1996; Gutkin and Aifantis, 1996) made extensive use of the latter type of nonlocal elasticity to solve, among others, crack-tip problems and showed that, whereas the strain field is smooth, the stress field still remains singular at the crack tip. In the following, only the former type of nonlocal elasticity will be considered and referred to without the word ‘integral’.

The purpose of the present paper is to address small-displacement small-strain elasticity theory with a nonlocal elastic material model as proposed by Eringen and co-workers (Eringen and Kim, 1974; Eringen et al., 1977; Eringen, 1978, 1979), referred to as the ‘Eringen model’ in the following. One of the main issues herein addressed is the thermodynamic framework of nonlocal elasticity. Many such frameworks have been envisaged in the literature, centered on the common idea of accounting for the nonlocality in elastic and plastic continua through the mentioned nonlocality residuals, Edelen and Laws (1971), Edelen et al. (1971), Eringen and Edelen (1972) and Eringen (1981, 1983). This idea is here followed only in writing the first principle (energy balance), whereas the second principle (internal entropy production inequality) is written in the classical pointwise form in consideration that the concept of reversible/irreversible material behavior possesses an essentially local nature. This procedure has been already applied by this writer and co-workers in the domain of (integral and gradient) nonlocal plasticity with satisfactory results (Polizzotto and Borino, 1998; Borino et al., 1999; Polizzotto et al., 2000).

The paper content is as follows. In Section 2 the Eringen model is reminded and refined by assuming the attenuation function to depend on the ‘geodetical distance’ between points, so enabling obstacles as holes and cracks not to be traversed in the diffusion processes of the nonlocality effects. The model’s features to guarantee positive definite strain energy are stated. In Section 3 the model is endowed with a firm basis within thermodynamics with nonlocality. This is achieved: (i) by adding, to the first principle written in pointwise form, a nonlocality energy residual term which accounts for the energy interchanges between the particles due to the nonlocality diffusion processes; (ii) by enforcing the second principle (entropy production rate inequality) in pointwise form to properly assess the irreversibility/reversibility features of the constitutive behavior. By this, the state equations, as well as the nonlocality residual, are evaluated. Section 4 is devoted to the boundary-value problem of nonlocal elasticity, whose solution (if any) is found to be unique due to the positive definiteness property of the strain energy. In Section 5, three variational principles of classical elasticity, that is, the total potential energy, the complementary energy and the Hu–Washizu principles, are extended to nonlocal elasticity. In Section 6, the former principle is used to derive a nonlocal-type FEM characterized by a solving equation system as for the standard FEM, but with a global stiffness matrix reflecting the problem's nonlocality features. In Section 7, an iterative procedure of the type local prediction/nonlocal correction is provided for use with the standard FEM, in which the nonlocality is injected through a suitable fictitious node force vector. Conclusions are drawn in the final Section 8. The notation used is given in Appendix A.

2. The nonlocal elastic material model

Let an aggregate of material particles as a continuum occupy the domain V in the three-dimensional Euclidean space and let every individual particle be referred to by the Cartesian orthogonal co-ordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ specifying its undisturbed position in V. In local elasticity, consideration of a single typical material particle would be sufficient in order to write out the stress–strain relations, whereas in the present case of nonlocal elasticity all the particles in the aggregate, which influence one another, must be considered to the same purpose.

2.1. The Eringen model

According to the Eringen model (see e.g. Eringen and Kim, 1974; Eringen et al., 1977; Eringen, 1978, 1979), the long range forces arising in a homogeneous isotropic elastic material as a consequence of a strain field, $\boldsymbol{\varepsilon}(\boldsymbol{x}) = \{\varepsilon_{ij}(\boldsymbol{x})\}$, are described by the stress field $\boldsymbol{\sigma}(\boldsymbol{x}) = \{\sigma_{ij}(\boldsymbol{x})\}$ given by

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \int_V A(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{D} : \boldsymbol{\varepsilon}(\boldsymbol{x}')\mathrm{d}V' \quad \forall \boldsymbol{x} \in V. \tag{1}$$

Here, $\mathrm{d}V' := \mathrm{d}V(\boldsymbol{x}')$ and $\boldsymbol{D} = \{D_{ijkh}\}$ denotes the elastic moduli fourth-rank tensor of classical isotropic elasticity, i.e.

$$D_{ijkh} = \lambda\delta_{ij}\delta_{kh} + \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}), \tag{2}$$

λ and μ being the Lamé constants and δ_{ij} the Kronecker symbol. The scalar function A is (after Eringen) the *attenuation function* related to the nonlocality effects of the (local) strain, a function of the Euclidean distance between the source ($\boldsymbol{x}'$) and field ($\boldsymbol{x}$) points, i.e. $A = a(r)$ with $r = |\boldsymbol{x}' - \boldsymbol{x}|$. $a(r)$, that is the attenuation function in terms of distance r, is nonnegative and decays more or less rapidly with increasing r, i.e. $a(r) \to 0$ for $r \to \infty$, but in practice $a(r) \simeq 0$ for $r \geqslant R$, where R is the (finite) *influence distance*. Eq. (1) reflects not only the material properties, but also the geometry of the domain occupied by the material in its natural state.

The material stress response $\boldsymbol{\sigma}(\boldsymbol{x})$ of Eq. (1) is referred to as the *nonlocal stress* to mean that it is a functional of the local strain $\boldsymbol{\varepsilon}$, that is, it is expressed as a weighted value of the strain field $\boldsymbol{\varepsilon}(\boldsymbol{x})$ over V. A noteworth feature of the nonlocal stress field, $\boldsymbol{\sigma}(\boldsymbol{x})$, is that its regularity degree is determined by that of the attenuation function $A(\boldsymbol{x}, \boldsymbol{x}')$ as function of $\boldsymbol{x}$, but not by the strain field $\boldsymbol{\varepsilon}(\boldsymbol{x})$, as it would occur in local elasticity.

An alternative form of Eq. (1) is:

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \mathscr{R}(\boldsymbol{s}) := \int_V A(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{s}(\boldsymbol{x}')\,\mathrm{d}V' \quad \forall \boldsymbol{x} \in V, \tag{3}$$

where $\boldsymbol{s} = \{s_{ij}\}$ is a (fictitious) local stress related to $\boldsymbol{\varepsilon}$ by the classical Hooke law, i.e.

$$\boldsymbol{s} = \boldsymbol{D} : \boldsymbol{\varepsilon}, \tag{4}$$

and $\mathscr{R}(\cdot)$ denotes a (linear) integral operator that transforms a local field, as $\boldsymbol{s}(\boldsymbol{x})$, into the related nonlocal one, as $\boldsymbol{\sigma}(\boldsymbol{x})$. It is worth noting that $\mathscr{R}$ is self-adjointed, that is a Green-type equality holds, i.e.

$$\int_V \mathscr{R}(G) \cdot F\,\mathrm{d}V = \int_V G \cdot \mathscr{R}(F)\,\mathrm{d}V, \tag{5}$$

identically for any pair of scalar or tensor fields, F and G, the scalar product being to be interpreted accordingly.

The attenuation characteristics of the attenuation function $a(r)$ make sense in relation to a parameter ℓ, the material internal length scale. Namely, a is in effect a function of the ratio $\rho := r/\ell$, i.e. $a = a(\rho)$, such that $a(\rho) \simeq 0$ for $\rho \gg 1$, but $a(\rho) \neq 0$ for $\rho \leqslant 1$, or of the same order of magnitude as 1. The distance r is 'large' or 'small' only relatively to ℓ. At small distances, the attenuation is moderate and the nonlocality effects can diffuse almost unaltered, whereas at large distances the attenuation is considerable and the nonlocality effects are sensibly reduced, till the complete arrest of their diffusion beyond the *influence distance* (i.e. the maximum distance within which a is meaningful) from the source points. At the macroscopic level, ℓ can be considered much smaller than the smallest dimension of the body or specimen. For $\ell \to 0$ the distance r is always 'large' and thus $a(r/\ell) = 0\ \forall r > 0$; but, since $a(0) \neq 0$, it results that the attenuation function a must correspondingly become a Dirac delta, i.e. $a(r/\ell) \to \Delta(r)$ for $\ell \to 0$, and in this case the nonlocal elasticity of Eq. (1) transforms into the local one, i.e. $\boldsymbol{\sigma} = \boldsymbol{D} : \boldsymbol{\varepsilon}$. The latter result can be achieved by requiring a to satisfy the condition

$$\int_{V_\infty} a(|\boldsymbol{x}' - \boldsymbol{x}|)\,\mathrm{d}V' = 1 \tag{6}$$

in which V_∞ is the (convex) infinite domain in which V is imbedded. Note that the value of the integral in Eq. (6) is independent of the field point $\boldsymbol{x}$ in V_∞; also, in Eq. (6) the dependence of a from ℓ has been disregarded for simplicity (as it will be done in the following).

Condition (6) deserves some additional comments. It guarantees that for the infinite domain, $V = V_\infty$, and in case of uniform strain, say $\boldsymbol{\varepsilon} = \bar{\boldsymbol{\varepsilon}}$ in V_∞, the related stress given by Eq. (1) is also uniform and complies with the local stress–strain relation, i.e. $\boldsymbol{\sigma} = \boldsymbol{D} : \bar{\boldsymbol{\varepsilon}}$ everywhere in V_∞. The physical interpretation of this result is that the diffusion processes of the nonlocality effects, promoted by the uniform strain $\bar{\boldsymbol{\varepsilon}}$, are allowed to be completely and normally accomplished in the absence of any impeding surface within V_∞. On the contrary, for a *finite* domain V with $\boldsymbol{\varepsilon} = \bar{\boldsymbol{\varepsilon}}$ in it, Eq. (1) gives

$$\boldsymbol{\sigma}(\boldsymbol{x}) = A_1(\boldsymbol{x})\boldsymbol{D} : \bar{\boldsymbol{\varepsilon}} \quad \forall \boldsymbol{x} \in V, \tag{7}$$

where

$$A_1(\boldsymbol{x}) := \int_V A(\boldsymbol{x}, \boldsymbol{x}')\,\mathrm{d}V'. \tag{8}$$

The value A_1 of the integral in Eq. (8) being variable with $\boldsymbol{x} \in V$, Eq. (7) shows that the nonlocal stress $\boldsymbol{\sigma}$ is nonuniform in V and that the locality of the stress cannot be gained in this case. The latter circumstance remains true even in the case (likely met in practice) that $a(r)$ is, or is considered, vanishing for any $r \geqslant R$ ($R > \ell$, but R relatively smaller than the smallest dimension of the domain), in which case $\boldsymbol{\sigma}(\boldsymbol{x})$ of Eq. (7)

turns out to be uniform within the core subdomain $V_C \subset V$, whose points $\boldsymbol{x}$ are located at a distance $r \geqslant R$ from ∂V, but $\boldsymbol{\sigma}$ is variable in the skin layer V/V_C. The motivation for the lack of uniformity and locality of $\boldsymbol{\sigma}$ in the considered case lies in the existence of an impeding boundary surface as ∂V, within which the (strain) source distribution and the consequent diffusion processes of the nonlocality effects must be confined.

Note that the condition $A_1(\boldsymbol{x}) = 1$, frequently invoked in the literature instead of Eq. (6), but with the same motivation accompanying Eq. (6), is not rigorously correct (it in fact can be satisfied for all $\boldsymbol{x} \in V$ if, and only if, A is a Dirac delta for whatsoever ℓ value, that is, only in case of local elasticity); however, it coincides with Eq. (6) for all $\boldsymbol{x} \in V_C$.

Typical choices for the attenuation function $a(r)$ are the following:

$$a := k \exp(-r^2/\ell^2) \quad \text{(error function)}, \tag{9a}$$

$$a := k\langle 1 - r^2/r_0^2 \rangle^2 \quad \text{(bell shape)}, \tag{9b}$$

$$a := k\langle 1 - r/r_0 \rangle \quad \text{(conical shape)}, \tag{9c}$$

where k is a constant to evaluate using Eq. (6), $\langle\cdot\rangle$ is the Macauley symbol, i.e. $\langle y \rangle = (y + |y|)/2$ for any value of the scalar y, and $r_0 = (m_0 + 1)\ell$, $m_0 = \text{integer} \geqslant 1$. The function a of Eq. (9a), the error function, has an unbounded support, whereas the functions in Eqs. (9b) and (9c) have a bounded support of length $2r_0$. The function a in Eq. (9a), which is indefinitely differentiable, as a rule will be considered in all subsequent developments. The plots of the functions $a(\rho)/k$ are depicted in Fig. 1.

As a constitutive equation of nonlocal elasticity, Eq. (1) must exhibit some essential features that are now pointed out. First, the functional relation $\boldsymbol{\sigma} = \boldsymbol{\sigma}[\boldsymbol{\varepsilon}]$ expressed by Eq. (1) must be invertible; in other words, Eq. (1), viewed as an integral equation, must give a unique strain field, $\boldsymbol{\varepsilon}(\boldsymbol{x})$, for any assigned stress field, $\boldsymbol{\sigma}(\boldsymbol{x})$. Second, the strain energy stored in V in any nontrivial strain state must be positive; that is, the integral inequality

$$W[\boldsymbol{\varepsilon}] := \int_V \int_V A(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{\varepsilon}(\boldsymbol{x}) : \boldsymbol{D} : \boldsymbol{\varepsilon}(\boldsymbol{x}')\,\mathrm{d}V'\,\mathrm{d}V > 0 \tag{10}$$

must be satisfied for any nontrivial $\boldsymbol{\varepsilon}(\boldsymbol{x})$ in V and thus $W[\boldsymbol{\varepsilon}] = 0$ can be satisfied only with $\boldsymbol{\varepsilon}$ identically vanishing in V.

Both the afore-mentioned requisites can be complied with by operating in the square summable functions (L_2) space – and this is the rule in this paper – as well as by assuming that the eigenvalue integral equation

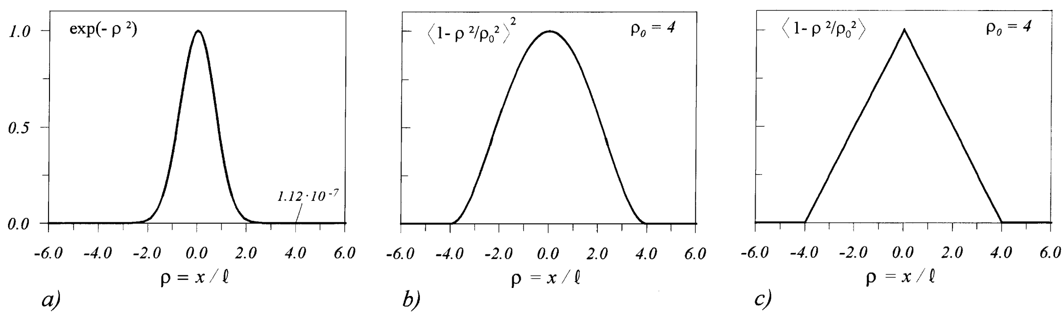

Fig. 1. Plots of the attenuation function $a(\rho)/k$ for positive and negative distance $\rho = x/\ell$ in a unidimensional space x: (a) error function; (b) bell shape function ($\rho_0 = 4$); (c) conical shape function ($\rho_0 = 2$).

$$\phi(\boldsymbol{x}) = \alpha \int_V A(\boldsymbol{x}, \boldsymbol{x}')\phi(\boldsymbol{x}')\,\mathrm{d}V' \tag{11}$$

is *nondegenerate*, the related eigenvalues $\{\alpha_1, \alpha_2, \ldots\}$ are all *positive* and the corresponding (ortho-normal) eigensolutions $\{\phi_1, \phi_2, \ldots\}$ constitute a *complete* set of functions in L_2. In fact, if this is the case, basing on the Fredholm integral equation theory (see e.g. Tricomi, 1985), one can state that the (symmetric) kernel $A(\boldsymbol{x}, \boldsymbol{x}')$ can be represented by the (uniformly convergent) series

$$A(\boldsymbol{x}, \boldsymbol{x}') = \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \phi_n(\boldsymbol{x})\phi_n(\boldsymbol{x}'), \tag{12}$$

and that any function of L_2 can be represented as a Fourier series, for example considering the function $\varepsilon_{ij}(\boldsymbol{x})$:

$$\varepsilon_{ij}(\boldsymbol{x}) = \sum_{n=1}^{\infty} E_{ij}^{(n)} \phi_n(\boldsymbol{x}), \tag{13}$$

where the $E_{ij}^{(n)}$ denote the related Fourier coefficients, i.e.

$$E_{ij}^{(n)} = \int_V \phi_n(\boldsymbol{x})\varepsilon_{ij}(\boldsymbol{x})\,\mathrm{d}V. \tag{14}$$

Therefore, for any strain field of L_2, represented as in Eq. (13), one obtains from Eq. (10), using Eq. (12),

$$W = \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \boldsymbol{E}^{(n)} : \boldsymbol{D} : \boldsymbol{E}^{(n)} > 0 \tag{15}$$

for arbitrary (nontrivial) tensor-valued Fourier coefficients $\boldsymbol{E}^{(n)}$.

The nondegenerateness of the eigenproblem (11) stems from the evident fact that the kernel A cannot be represented with a series like Eq. (12), but a finite number of terms. The completeness of the set of eigenfunctions $\{\phi_1, \phi_2, \ldots\}$ is a consequence of the fact that the equation

$$\int_V A(\boldsymbol{x}, \boldsymbol{x}')\phi(\boldsymbol{x}')\,\mathrm{d}V' = 0 \quad \forall \boldsymbol{x} \in V \tag{16}$$

with the kernel A being the previously considered attenuation function, as a consequence of the assumption (10), admits only the trivial solution $\phi = 0$ in L_2, and this fact, by a theorem of the integral equation theory (Tricomi, 1985), implies that the only function of L_2 normal to every eigenfunction ϕ_n is the null function. Finally, the positiveness of the eigenvalues $\{\alpha_1, \alpha_2, \ldots\}$ is at present only an assumption, likely met in all practical situations, but a stronger statement would be hoped for to this concern.

This section is concluded by observing that often in the literature (see e.g. Eringen, 1987; Altan, 1989a) the constitutive equation for nonlocal elasticity is set in the form

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \xi_1 \boldsymbol{D} : \boldsymbol{\varepsilon}(\boldsymbol{x}) + \xi_2 \int_V A(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{D} : \boldsymbol{\varepsilon}(\boldsymbol{x}')\,\mathrm{d}V', \tag{17}$$

where ξ_1 and ξ_2 are positive material constants. Assuming $\xi_1 + \xi_2 = 1$, Eq. (17) can be read as the constitutive equation of a two-phase elastic material, in which phase 1 (of volume fraction ξ_1) complies with local elasticity, phase 2 (of volume fraction ξ_2) complies with nonlocal elasticity. For A being a Dirac delta, phase 2 becomes a local-elasticity phase like phase 1 and thus the two-phase model turns out to be equivalent to the classical model of local elasticity. Taking a modified attenuation function as

$$\bar{A} := \xi_1 \Delta(\boldsymbol{x}' - \boldsymbol{x}) + \xi_2 A(\boldsymbol{x}, \boldsymbol{x}'), \tag{18}$$

where $\Delta(\boldsymbol{x}' - \boldsymbol{x})$ is the Dirac delta centered at the field point $\boldsymbol{x}$, Eq. (17) can be rewritten as

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \int_V \bar{A}(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{D} : \boldsymbol{\varepsilon}(\boldsymbol{x}')\,\mathrm{d}V', \tag{19}$$

which is of the same form as Eq. (1), though the kernel $\bar{A}$ is weakly singular. Thus, the stress–strain relation (17) can be considered a special case of Eq. (1).

2.2. A refined model

The attenuation function $A = a(|\boldsymbol{x}' - \boldsymbol{x}|)$ in Eq. (1) interprets the *diffusion law* by which the nonlocality effects propagate in the body's domain. After this law, the diffusion takes place along straight paths from the source points to all others, even in the case that the straight path traverses a crack, or a hole, that may exist in the body. It is reasonable to conjecture that cracks and holes in the domain, and more in general any gap in the convexity of the domain, represent obstacles for the diffusion processes which consequently cannot follow any straight path joining two particles, that traverses the boundary surface ∂V. In the absence of experimental data, this difficulty can be overcome in a quite natural way by conjecturing that the diffusion processes of the nonlocality effects follow *paths of minimum length not intersecting the boundary surface of the body*, or (as they will be referred to in this paper) *geodetic paths*. This implies that the distance r of any pair $(\boldsymbol{x}, \boldsymbol{x}') \in V$ on which the attenuation function $a(r)$ actually depends is the related geodetical distance, say $r(\boldsymbol{x}, \boldsymbol{x}')$, not the Euclidean distance $|\boldsymbol{x}' - \boldsymbol{x}|$. Note that, for a given pair $(\boldsymbol{x}, \boldsymbol{x}') \in V$, the geodetical distance is unique – though the related geodetical path may sometimes be nonunique. Also note that $r(\boldsymbol{x}, \boldsymbol{x}') \geqslant \mid \boldsymbol{x}' - \boldsymbol{x} \mid$ for any pair $(\boldsymbol{x}, \boldsymbol{x}')$ and that the equality sign holds if, and only if, the straight path joining the two points does not traverse ∂V (this is always the case if the domain is convex, hence with no cracks and holes, nor re-entrant angles). With the above conjecture in mind, Eq. (1) with the attenuation function depending on the geodetical distance. i.e. with $A(\boldsymbol{x}, \boldsymbol{x}') = a(r(\boldsymbol{x}, \boldsymbol{x}'))$, constitutes a refined model of nonlocal elasticity, able to account for the effects produced by the domain's nonconvexity on the mentioned diffusion processes.

A reasoning to support the physical – from the macroscopic point of view – consistency of the geodetical distance in relation to the nonlocality diffusion law is the following. If a body of domain V, with its (refined) constitutive equation (1), is cut in two distinct portions, say V_1 and V_2, considered that the geodetical distance between any point $\boldsymbol{x}_1 \in V_1$ from any point $\boldsymbol{x}_2 \in V_2$ is infinite, and thus $A(\boldsymbol{x}_1, \boldsymbol{x}_2) = 0$, Eq. (1) naturally splits in two distinct constitutive equations, i.e.

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \int_{V_1} A(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{D} : \boldsymbol{\varepsilon}(\boldsymbol{x}')\,\mathrm{d}V' \quad \forall \boldsymbol{x} \in V_1, \tag{20a}$$

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \int_{V_2} A(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{D} : \boldsymbol{\varepsilon}(\boldsymbol{x}')\mathrm{d}V' \quad \forall \boldsymbol{x} \in V_2. \tag{20b}$$

This is in accordance with the nature of the considered nonlocality effects, which in fact cannot migrate from a body to another distinct one.

The latter result can be obtained even if the cut is operated progressively, as illustrated in Fig. 2(a)–(c). The convex body of domain V in Fig. 2(a) is characterized by a constitutive equation as Eq. (1), with $r = |\boldsymbol{x}' - \boldsymbol{x}|$ because of the convexity. In Fig. 2(b), the same body has an incision somewhere within V on the (internal) surface Γ, but the body remains unique: Eq. (1) still applies no matter how wide the incision, but with changes to account for the geometry changes (the straight paths intersecting Γ are replaced by geodetical paths circumventing it). In Fig. 2(c) the incision is so wide as to split the body in two distinct portions, in which case Eq. (1) is known to split in two distinct equations, like Eqs. (20a) and (20b). Had the distance $r = r(\boldsymbol{x}, \boldsymbol{x}')$ to be meant as the Euclidean distance, Eq. (1) would remain unchanged for all

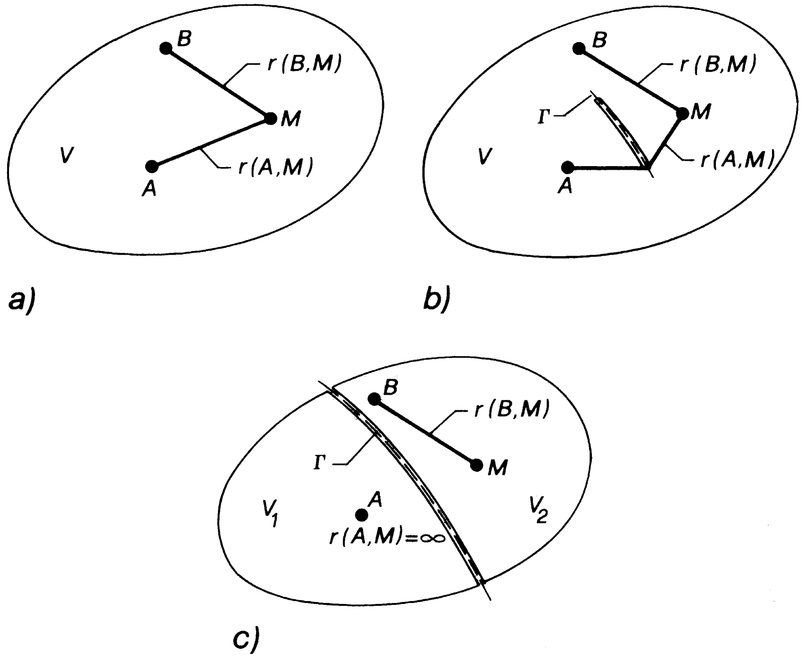

Fig. 2. Nonlocal body with convex domain V and crack in progress: (a) no crack in V, then the geodetical distance coincides with the Euclidean distance, i.e. $r(A,M) = \overline{AM}$ for all points A, M in V; (b) V has a crack, there exist points as A, M in V whose geodetical distance $r(A,M) > \overline{AM}$; (c) a wide crack splits V in two portions, V_1 and V_2, then $r(A,M) = \infty$ for $A \in V_1$ and $M \in V_2$ (no path from A, no matter how long, is allowed to traverse the boundary of V_1 to reach M in V_2).

situations described by Fig. 2(a)–(c) (actually occurring in a real cracking process), what obviously is unacceptable.

The Euclidean distance as argument of the attenuation function was introduced (see e.g. Kröner, 1967; Eringen, 1972a, 1987) for a twofold reason: (i) the invariance of Eq. (1) against rigid motions of the body; (ii) the isotropy of the material response (i.e. the nonlocality effects propagate equally in all directions from the source point). One can observe, correspondingly, that: (i) the invariance of Eq. (1) is preserved on replacing the Euclidean distance with the geodetical distance; (ii) the nonlocality effects propagate equally in all directions from the source point as long as no lack of material continuity is encountered along the straight paths (like a crack or a hole), in which case a sort of geometric anisotropy occurs – and is actually accounted for by the geodetical distance concept.

All what has been established in Section 2.1 remains unaltered if r is meant as geodetical distance instead of the Euclidean distance. In particular, Eq. (6) holds good because $r = |\boldsymbol{x}' - \boldsymbol{x}|$, hence $A = a(|\boldsymbol{x}' - \boldsymbol{x}|)$, for any $(\boldsymbol{x}, \boldsymbol{x}') \in V_\infty$. As a rule, Eq. (1) in its refined version will be employed in the following developments.

3. Thermodynamic framework

The nonlocal elastic model presented in Section 2 can be cast within a thermodynamic framework as in the following. Let, by assumption, an internal energy density potential, say $e = e(\boldsymbol{\varepsilon}, \mathscr{R}(\boldsymbol{\varepsilon}), \eta)$, there exist for the nonlocal continuum introduced in Section 2, where η denotes the entropy and $\mathscr{R}$ is the integral operator of Section 2. By hypothesis, e is a C^2-continuous function of its arguments; also, the internal energy at a point $\boldsymbol{x} \in V$ turns out to be a functional of the strain field $\boldsymbol{\varepsilon}$, but a function of the entropy at the same point. This means that the thermal diffusion processes in the nonlocal continuum are, by hypothesis, of local nature. Such a functional dependence of e on $\boldsymbol{\varepsilon}$ implies the existence in V of some diffusion processes of the nonlocality effects, promoted by the strain field at the microstructure level, by which long range cohesive stresses are induced along with energy exchanges between the material particles. These diffusion processes, in turn, imply that the principle of local action of thermodynamics does not hold and thus the first principle of thermodynamics can be written only for the whole body, i.e. (see e.g. Germain et al., 1983; Lemaitre and Chaboche, 1990):

$$\int_V \dot{e}\,\mathrm{d}V = \int_V (\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + h - \operatorname{div}\boldsymbol{q})\,\mathrm{d}V, \tag{21}$$

where h is the heat source per unit volume and $\boldsymbol{q}$ the heat conduction vector.

However, following Edelen and Laws (1971), Edelen et al. (1971), Eq. (21) can be written in pointwise form as

$$\dot{e} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + h - \operatorname{div}\boldsymbol{q} + P \quad \text{in } V, \tag{22}$$

where P is the *nonlocality* (*energy*) *residual*. P represents the energy transmitted to a material particle by all other particles in V through the mentioned nonlocality diffusion processes. As will be shown in this section, the constitutive dependence of P is as $P = P(\boldsymbol{\varepsilon}, \mathscr{R}(\boldsymbol{\varepsilon}), T, \dot{\boldsymbol{\varepsilon}}, \mathscr{R}(\dot{\boldsymbol{\varepsilon}}))$, where T is the absolute temperature. The following *insulation condition* holds for P, i.e.

$$\int_V P\,\mathrm{d}V = 0, \tag{23}$$

which expresses the fact that the nonlocality diffusion processes are confined within ∂V.

Let $\dot{\eta}_{\text{int}}$ denote the internal entropy production density rate, defined as

$$\dot{\eta}_{\text{int}} := \dot{\eta} - \left[\frac{h}{T} - \operatorname{div}\left(\frac{\boldsymbol{q}}{T}\right)\right]. \tag{24}$$

In the framework of local continua, the second principle of thermodynamics ultimately states that the internal entropy production density rate is nonnegative, i.e. $\dot{\eta}_{\text{int}} \geqslant 0$, at every point in V for all thermo-mechanical deformation processes, with the equality sign holding only for reversible ones. Such a local form of the second principle is quite reasonable as in fact, were the second principle valid only in the global form $\int_V \dot{\eta}_{\text{int}}\,\mathrm{d}V \geqslant 0$, there would exist thermo-mechanical deformation processes for which $\int_V \dot{\eta}_{\text{int}}\,\mathrm{d}V = 0$, that is, reversible at the global level but not at the local one, which obviously is physically meaningless. This remains true also for nonlocal continua. Indeed, the concepts of reversibility and irreversibility for the thermo-mechanical deformation processes seem to possess an essentially local nature. Thus, at difference with other authors (e.g. Edelen and Laws, 1971), the second principle of thermodynamics will be applied in its local form, i.e. $\dot{\eta}_{\text{int}} \geqslant 0$ in V, in the following.

Then, introducing the Helmholtz free energy $\psi = \psi(\boldsymbol{\varepsilon}, \mathscr{R}(\boldsymbol{\varepsilon}), T)$ through the Legendre transform, i.e. $\psi = e - T\eta$, and using Eq. (24) in connection with the second principle, Eq. (22) can be rewritten as

$$T\dot{\eta}_{\text{int}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} - \eta\dot{T} + \Phi_{\text{T}} + P \geqslant 0 \quad \text{in } V \tag{25}$$

where Φ_{T} is the energy dissipation by thermal diffusion, given by

$$\Phi_{\text{T}} = -\nabla T \cdot \frac{\boldsymbol{q}}{T}. \tag{26}$$

Inequality (25) is the Clausius–Duhem inequality, differing from its classical form for the presence of the residual P (but the latter disappears on integrating Eq. (25) over V due to Eq. (23)).

Inequality (25) can be used to derive the thermodynamic restrictions to the constitutive equations for nonlocal elasticity, following to this purpose a well established procedure by Coleman and Gurtin (1967) (see also Germain et al., 1983; Lemaitre and Chaboche, 1990). Since inequality (25) must be satisfied by all thermo-mechanical deformation processes, let one first consider thermo-elastic processes at uniform temperature, i.e. $\nabla T = \mathbf{0}$, hence $\Phi_{\mathrm{T}} = 0$ in V, for which Eq. (25) simplifies as

$$T\dot{\eta}_{\mathrm{int}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} - \eta\dot{T} + P \geqslant 0 \quad \text{in } V. \tag{27}$$

Then, on developing the time derivative of ψ, integrating over V and taking account of Eq. (23), Eq. (27) gives

$$\int_V \left[\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial\psi}{\partial\mathscr{R}(\boldsymbol{\varepsilon})} : \mathscr{R}(\dot{\boldsymbol{\varepsilon}}) - \left(\eta + \frac{\partial\psi}{\partial T}\right)\dot{T}\right]\mathrm{d}V \geqslant 0, \tag{28}$$

which, by the Green identity (5), is equivalent to

$$\int_V \left[\boldsymbol{\sigma} - \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} - \mathscr{R}\left(\frac{\partial\psi}{\partial\mathscr{R}(\boldsymbol{\varepsilon})}\right)\right] : \dot{\boldsymbol{\varepsilon}}\,\mathrm{d}V - \int_V \left(\eta + \frac{\partial\psi}{\partial T}\right)\dot{T}\,\mathrm{d}V \geqslant 0. \tag{29}$$

As the latter inequality must hold for all thermo-elastic processes, hence for arbitrary choices of the fields $\dot{\boldsymbol{\varepsilon}}$ and $\dot{T}$, one obtains as necessary and sufficient conditions of Eq. (29):

$$\boldsymbol{\sigma} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} + \mathscr{R}\left(\frac{\partial\psi}{\partial\mathscr{R}(\boldsymbol{\varepsilon})}\right) \quad \text{in } V, \tag{30}$$

$$\eta = -\frac{\partial\psi}{\partial T} \quad \text{in } V, \tag{31}$$

which are the state equations. These imply that Eqs. (29) and (28) hold as equalities and thus, because Eq. (28) is the volume integral of a nonnegative quantity, i.e. the left-hand side of Eq. (27), follows that Eq. (27) too holds as an equality, i.e.

$$T\dot{\eta}_{\mathrm{int}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} - \eta\dot{T} + P = 0 \quad \text{in } V. \tag{32}$$

This reveals the reversible nature of the considered processes and enables the constitutive equation of P to be evaluated, i.e.

$$P = \frac{\partial\psi}{\partial\mathscr{R}(\boldsymbol{\varepsilon})} : \mathscr{R}(\dot{\boldsymbol{\varepsilon}}) - \mathscr{R}\left(\frac{\partial\psi}{\partial\mathscr{R}(\boldsymbol{\varepsilon})}\right) : \dot{\boldsymbol{\varepsilon}} \quad \text{in } V. \tag{33}$$

Then, assuming that Eqs. (30)–(33) remain valid also in the more general case $\nabla T \not\equiv 0$, Eq. (25) simplifies as

$$T\dot{\eta}_{\mathrm{int}} = \Phi_{\mathrm{T}} \geqslant 0 \quad \text{in } V, \tag{34}$$

which expresses the nonnegativity of Φ_{T}.

An interesting interpretation of the role played by the nonlocality residual P deserves being pointed out. To this purpose, let one consider isothermal elastic deformation processes ($\dot{T} = 0$), for which Eq. (32) reads

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + P = \dot{\psi}|_{T=\mathrm{const}} \quad \text{in } V \tag{35}$$

and thus, by an integration over V and using Eq. (23),

$$\int_V \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}\,\mathrm{d}V = \int_V \dot{\psi}|_{T=\mathrm{const}}\,\mathrm{d}V. \tag{36}$$

The latter equality means that the total energy from mechanical work done against the particle system transforms into total free energy, just as it would occur in local elasticity. But such a balance between mechanical work and free energy is no longer true at the local level since in fact, as stated by Eq. (35), the total energy imparted to the system redistributes within V with additions at points where $P > 0$ (particles there located receive additional energy, P, from other particles through the nonlocality diffusion processes), and losses at points where $P < 0$ (particles there located yield energy, $-\mid P\mid$, to the benefit of other particles in V).

Eq. (30) is the stress–strain relation of a nonlocal hyperelastic material endowed with strain energy ψ; it can be given more precise expressions by making suitable choices for ψ. For instance, assuming $T = \text{constant}$, let ψ be taken linear in the operator $\mathscr{R}$ as

$$\psi = \tfrac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{D}_1 : \boldsymbol{\varepsilon} + \tfrac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{D}_2 : \mathscr{R}(\boldsymbol{\varepsilon}), \tag{37}$$

where $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are moduli tensors. Then, Eq. (30) gives

$$\boldsymbol{\sigma} = \boldsymbol{D}_1 : \boldsymbol{\varepsilon} + \boldsymbol{D}_2 : \mathscr{R}(\boldsymbol{\varepsilon}) \tag{38}$$

which for $\boldsymbol{D}_1 = \xi_1\boldsymbol{D}$ and $\boldsymbol{D}_2 = \xi_2\boldsymbol{D}$ coincides with Eq. (17).

The above developments can be easily extended to nonlocal thermoelasticity, that is, nonlocal elasticity coupled with a nonlocal thermal diffusion law, but this issue is not pursued here. In the following developments isothermal processes will be considered with the (refined) stress–strain relation (1).

4. The boundary-value problem

Let the same aggregate of material particles of Section 2 be considered again, with a constitutive behavior of nonlocal elasticity as therein described. This aggregate is the material of a continuous solid body which, in its undeformed state, occupies the (open) domain V. The body is subjected to external actions as volume forces $\bar{\boldsymbol{b}}(\boldsymbol{x})$ in V, surface forces $\bar{\boldsymbol{t}}(\boldsymbol{x})$ on the portion S_t of the boundary surface ∂V, and imposed displacements $\bar{\boldsymbol{u}}(\boldsymbol{x})$ on S_u, the constrained portion of ∂V. By hypothesis, all these data fields are sufficiently smooth in their respective domains. (No imposed thermal-like strains are considered for simplicity, but they may be easily included.)

On application, in a quasi-static manner, of the above loads to the body being in its initial undeformed state, the body's response can be obtained by solving the following system of equations:

$$\operatorname{div}\boldsymbol{\sigma} + \bar{\boldsymbol{b}} = 0 \quad \text{in } V \ \text{(field equilibrium)}, \tag{39a}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \bar{\boldsymbol{t}} \quad \text{on } S_t \ \text{(boundary equilibrium)}, \tag{39b}$$

$$\boldsymbol{\varepsilon} = \nabla^{\mathrm{s}}\boldsymbol{u} \quad \text{in } V \ \text{(field compatibility)}, \tag{39c}$$

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \quad \text{on } S_u \ \text{(boundary compatibility)}, \tag{39d}$$

$$\boldsymbol{\sigma} = \boldsymbol{D} : \mathscr{R}(\boldsymbol{\varepsilon}) \quad \text{in } V \ \text{(nonlocal elasticity)}. \tag{39e}$$

Here, 'div' is the divergence operator, i.e. $\operatorname{div}\boldsymbol{\sigma} = \{\sigma_{ji,j}\}$; ∇^{s} = symmetric part of the gradient operator, i.e. $\nabla^{\mathrm{s}}\boldsymbol{u} = \{(u_{i,j} + u_{j,i})/2\}$; $\boldsymbol{n} = \{n_i\}$ = unit normal external vector to $\partial V = S_u \cup S_t S_u \cap S_t = 0$; the overposed bar on a quantity means assigned value of this quantity.

The equation set (39a)–(39e) constitutes an analysis problem for linear (homogeneous isotropic) nonlocal elasticity. Using the constitutive properties of the material as established, or assumed, in Section 2, the uniqueness of the solution (if it exists) of the above analysis problem can be proved through a classical procedure. Namely, let by absurdity there exist two distinct solutions, say $\boldsymbol{u}'$, $\boldsymbol{\varepsilon}'$, $\boldsymbol{\sigma}'$ and $\boldsymbol{u}''$, $\boldsymbol{\varepsilon}''$, $\boldsymbol{\sigma}''$. As the difference strain field, $\Delta\boldsymbol{\varepsilon} := \boldsymbol{\varepsilon}' - \boldsymbol{\varepsilon}''$, is compatible with zero initial strains in V and zero displacements on S_u, and as the difference stress field, $\Delta\boldsymbol{\sigma} := \boldsymbol{\sigma}' - \boldsymbol{\sigma}''$, is in equilibrium with zero volume forces in V and zero surface forces on S_t, the virtual work principle yields:

$$\int_V \Delta\boldsymbol{\sigma} : \Delta\boldsymbol{\varepsilon}\,\mathrm{d}V = 0. \tag{40}$$

Then, since $\Delta\boldsymbol{\sigma} = \boldsymbol{D} : \mathscr{R}(\Delta\boldsymbol{\varepsilon})$ by Eq. (39e), using Eq. (1), Eq. (40) turns out to be equivalent to

$$\int_V \int_V A(\boldsymbol{x}, \boldsymbol{x}')\Delta\boldsymbol{\varepsilon}(\boldsymbol{x}) : \boldsymbol{D} : \Delta\boldsymbol{\varepsilon}(\boldsymbol{x}')\,\mathrm{d}V'\,\mathrm{d}V = 0 \tag{41}$$

which can be satisfied if and only if, $\Delta\boldsymbol{\varepsilon} = \boldsymbol{0}$, i.e. $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}''$, hence $\boldsymbol{\sigma}' = \boldsymbol{\sigma}''$, everywhere in V; thus, as long as $S_u \neq \emptyset$, it is $\boldsymbol{u}' = \boldsymbol{u}''$ in $V \cup \partial V$. An analogous proof of solution uniqueness, based on the positive definiteness of the Hilbert quadratic form (10), was given by Altan (1989a).

It is worth of note that the regularity degree demanded to the displacement field $\boldsymbol{u}(\boldsymbol{x})$ in problem (39a)–(39e) is milder than in the analogous problem of local elasticity, as in fact in the present case the stress field $\boldsymbol{\sigma}(\boldsymbol{x})$ given by Eq. (39e) is as regular as the attenuation function, and this independently of the regularity degree of $\boldsymbol{u}(\boldsymbol{x})$, the latter being allowed to be C^1-continuous in V in nonlocal elasticity. This has two consequences: (i) the solution to the nonlocal elasticity problem may not exist for volume and surface forces being not sufficiently regular; and (ii) it does not exhibit the typical singularity of the stress field as encountered in classical local elasticity in the presence of a sharp crack or similar geometry irregularities (Eringen, 1978, 1979).

The nonlocal elasticity problem (39a)–(39e) can be shown to admit variational formulations analogous to those of classical local elasticity. This is the subject of next section.

5. Variational principles

In this section, three variational principles are presented. They are extensions to nonlocal elasticity of, respectively, the total potential energy, the complementary energy and the mixed Hu–Washizu principles of classical (local) elasticity (Washizu, 1982).

5.1. Total potential energy principle for nonlocal elasticity

The total potential energy pertaining to the body of nonlocal elastic material as considered in Section 4 is expressed by the following functional $\Pi[\boldsymbol{u}]$, i.e.

$$\Pi[\boldsymbol{u}] := \frac{1}{2}\int_V \int_V A(\boldsymbol{x}, \boldsymbol{x}')\nabla^{\mathrm{s}}\boldsymbol{u}(\boldsymbol{x}) : \boldsymbol{D} : \nabla^{\mathrm{s}}\boldsymbol{u}(\boldsymbol{x}')\mathrm{d}V'\,\mathrm{d}V - \int_V \bar{\boldsymbol{b}} \cdot \boldsymbol{u}\,\mathrm{d}V - \int_{S_t} \bar{\boldsymbol{t}} \cdot \boldsymbol{u}\,\mathrm{d}S, \tag{42}$$

where the field $\boldsymbol{u}(\boldsymbol{x})$ belongs to the class $\mathscr{K}$ of all *kinematically admissible* displacement fields defined as follows: every $\boldsymbol{u}$ is C^1-continuous (it thus belongs to L_2 together with its first partial derivatives) and moreover satisfies the boundary compatibility condition (39d). The following can be proved:

Theorem 1. *The displacement field* $\boldsymbol{u} \in \mathscr{K}$, *which is part of the solution* $(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ *of the nonlocal elasticity problem* (39a)–(39e), *minimizes the total potential energy Eq.* (42) *in the class* $\mathscr{K}$; *conversely*, *the field* $\boldsymbol{u}$ *minimizing* (42) *in* $\mathscr{K}$ *is part of the solution of the nonlocal elasticity problem.*

Proof of the first part of the theorem. Let $\boldsymbol{u}^* \in \mathscr{K}$ be the displacement field pertaining to the (unique) solution (by hypothesis existing) to problem (39a)–(39e) and let $\boldsymbol{\varepsilon}^*$, $\boldsymbol{\sigma}^*$ be the related strains and stresses. Also, let $\boldsymbol{u} = \boldsymbol{u}^* + \delta\boldsymbol{u}$ be any other kinematically admissible displacement field, such that the variation fields $\delta\boldsymbol{u} \in \mathscr{K}'$, where $\mathscr{K}'$ is like $\mathscr{K}$ but $\bar{\boldsymbol{u}} = 0$ on S_u. From Eq. (42) one has:

$$\Pi[\boldsymbol{u}] = \Pi[\boldsymbol{u}^* + \delta\boldsymbol{u}] = \Pi[\boldsymbol{u}^*] + \delta\Pi + \tfrac{1}{2}\delta^2\Pi, \tag{43}$$

where $\delta\Pi$ and $\delta^2\Pi$ are the first and second variations of Π from $\boldsymbol{u}^*$. $\delta\Pi$ reads as follows:

$$\delta\Pi = \frac{1}{2}\int_V \int_V A(\boldsymbol{x}, \boldsymbol{x}')\left[\boldsymbol{\varepsilon}^*(\boldsymbol{x}) : \boldsymbol{D} : \delta\boldsymbol{\varepsilon}(\boldsymbol{x}') + \delta\boldsymbol{\varepsilon}(\boldsymbol{x}) : \boldsymbol{D} : \boldsymbol{\varepsilon}^*(\boldsymbol{x}')\right]\mathrm{d}V'\,\mathrm{d}V - \int_V \bar{\boldsymbol{b}} \cdot \delta\boldsymbol{u}\,\mathrm{d}V - \int_{S_t} \bar{\boldsymbol{t}} \cdot \delta\boldsymbol{u}\,\mathrm{d}S, \tag{44}$$

which is equivalent to

$$\delta\Pi = \int_V \delta\boldsymbol{\varepsilon}(\boldsymbol{x}) : \int_V A(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{D} : \boldsymbol{\varepsilon}^*(\boldsymbol{x}')\,\mathrm{d}V'\,\mathrm{d}V - \int_V \bar{\boldsymbol{b}} \cdot \delta\boldsymbol{u}\,\mathrm{d}V - \int_{S_t} \bar{\boldsymbol{t}} \cdot \delta\boldsymbol{u}\,\mathrm{d}S = \int_V \boldsymbol{\sigma}^* : \delta\boldsymbol{\varepsilon}\,\mathrm{d}V - \int_V \bar{\boldsymbol{b}} \cdot \delta\boldsymbol{u}\,\mathrm{d}V - \int_{S_t} \bar{\boldsymbol{t}} \cdot \delta\boldsymbol{u}\,\mathrm{d}S. \tag{45}$$

Since $\delta\boldsymbol{\varepsilon} = \nabla^s(\delta\boldsymbol{u})$ in V and $\delta\boldsymbol{u} = 0$ on S_u, and since $\boldsymbol{\sigma}^*$ is sufficiently regular, the divergence theorem can be applied to obtain:

$$\delta\Pi = -\int_V (\operatorname{div}\boldsymbol{\sigma}^* + \bar{\boldsymbol{b}}) \cdot \delta\boldsymbol{u}\,\mathrm{d}V + \int_{S_t} (\boldsymbol{n} \cdot \boldsymbol{\sigma}^* - \bar{\boldsymbol{t}}) \cdot \delta\boldsymbol{u}\,\mathrm{d}S, \tag{46}$$

and thus $\delta\Pi = 0$ by Eqs. (39a) and (39b). Furthermore,

$$\delta^2\Pi = \int_V \int_V A(\boldsymbol{x}, \boldsymbol{x}')\delta\boldsymbol{\varepsilon}(\boldsymbol{x}) : \boldsymbol{D} : \delta\boldsymbol{\varepsilon}(\boldsymbol{x}')\,\mathrm{d}V'\,\mathrm{d}V. \tag{47}$$

Considering that $\delta^2\Pi > 0$ for any nontrivial $\delta\boldsymbol{u} \in \mathscr{K}'$, Eq. (43) yields:

$$\Pi[\boldsymbol{u}] = \Pi[\boldsymbol{u}^*] + \tfrac{1}{2}\delta^2\Pi \geqslant \Pi[\boldsymbol{u}^*] \quad \forall \boldsymbol{u} \in \mathscr{K} \tag{48}$$

with the equality sign holding if and only if, $\boldsymbol{u} = \boldsymbol{u}^*$ everywhere in V. □

Proof of the second part of the theorem. Let $\boldsymbol{u}^*(\boldsymbol{x})$ be the/a solution (assumed to exist) to the minimum problem

$$\min_{(\boldsymbol{u})} \Pi[\boldsymbol{u}] \quad \text{s.t. } \boldsymbol{u} \in \mathscr{K}, \tag{49}$$

where 's.t.' means 'subject to'. This implies that the first variation of Π computed at $\boldsymbol{u}^*$, $\delta\Pi$ say, must vanish for any displacement variation $\delta\boldsymbol{u} \in \mathscr{K}'$. Since this $\delta\Pi$ reads as in Eq. (46), but with $\boldsymbol{u}^*$ having the present new meaning, it results that the stress field $\boldsymbol{\sigma}^* := \boldsymbol{D} : \mathscr{R}(\nabla^s\boldsymbol{u}^*)$ satisfies the equilibrium equations (39a) and (39b), and thus $\boldsymbol{u}^*$, $\boldsymbol{\varepsilon}(\boldsymbol{u}^*)$ and $\boldsymbol{\sigma}(\boldsymbol{u}^*)$ solve the nonlocal elasticity problem (and problem (49) has unique solution). The proof is so completed. □

Problem (39a)–(39e) transforms into the classical local-type boundary-value problem when the nonlocal elasticity Eq. (39e) is substituted with Hooke's law, i.e. $\boldsymbol{\sigma} = \boldsymbol{D} : \boldsymbol{\varepsilon}$ in V. It is worth remarking that the displacement solution to the latter problem, say $\boldsymbol{u}_0(\boldsymbol{x})$, is kinematically admissible, i.e. $\boldsymbol{u}_0 \in \mathscr{K}$, and thus $\Pi[\boldsymbol{u}_0] > \Pi[\boldsymbol{u}^*]$, $\boldsymbol{u}^*$ being the nonlocal displacement solution.

5.2. Complementary energy principle for nonlocal elasticity

Making still reference to the analysis problem (39a)–(39e), let the constitutive equation (39e) be replaced by Eqs. (3) and (4), here rewritten for more clarity, i.e.

$$\boldsymbol{\sigma} = \mathscr{R}(\boldsymbol{s}), \quad \boldsymbol{s} = \boldsymbol{D} : \boldsymbol{\varepsilon} \qquad \text{in } V. \tag{50}$$

In this way the solution to the above problem is described in terms of the fields in the set ($\boldsymbol{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$, $\boldsymbol{s}$), one more than previously, the local stress field $\boldsymbol{s}(\boldsymbol{x})$.

The complementary potential energy of the body is the functional:

$$\Pi_c[\boldsymbol{s}] := \frac{1}{2}\int_V \int_V A(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{s}(\boldsymbol{x}) : \boldsymbol{D}^{-1} : \boldsymbol{s}(\boldsymbol{x}')\,\mathrm{d}V'\,\mathrm{d}V - \int_{S_u}\int_V A(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{s}(\boldsymbol{x}') \cdot \bar{\boldsymbol{u}}(\boldsymbol{x})\,\mathrm{d}V'\,\mathrm{d}S, \tag{51}$$

where $\boldsymbol{s}(\boldsymbol{x})$ belongs to the class $\mathscr{H}$ of all *statically admissible* (local) stress fields defined as: every $\boldsymbol{s}$ is C^0-continuous (it thus belongs to L_2) and the related nonlocal stress, $\boldsymbol{\sigma} = \mathscr{R}(\boldsymbol{s})$, complies with field and boundary equilibrium, Eqs. (39a) and (39b). The following can then be proved:

Theorem 2. *The local stress field* $\boldsymbol{s} \in \mathscr{H}$, *part of the* (*unique*) *solution* ($\boldsymbol{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$, $\boldsymbol{s}$) *of the nonlocal elasticity problem* (39a)–(39d) *and* Eq. (50), *minimizes the complementary potential energy* (51) *in* $\mathscr{H}$; *conversely, the local stress field minimizing* Eq. (51) *is part of the solution to the same problem.*

Proof of the first part of the theorem. Let $\boldsymbol{u}^*$, $\boldsymbol{\varepsilon}^*$, $\boldsymbol{\sigma}^*$, $\boldsymbol{s}^*$ denote the (unique) solution (assumed to exist) to the nonlocal elasticity problem. Obviously, $\boldsymbol{s}^* \in \mathscr{H}$, whereas all other statically admissible stress fields can be written as $\boldsymbol{s} = \boldsymbol{s}^* + \delta\boldsymbol{s}$, where $\delta\boldsymbol{s}$ denotes (local) stress variation fields in the class $\mathscr{H}'$ similar to $\mathscr{H}$, but $\bar{\boldsymbol{b}} = 0$ in V and $\bar{\boldsymbol{t}} = 0$ on S_t (i.e. $\delta\boldsymbol{\sigma} := \mathscr{R}(\delta\boldsymbol{s})$ is a self-stress field). Eq. (51) then gives:

$$\Pi_c[\boldsymbol{s}] = \Pi_c[\boldsymbol{s}^*] + \delta\Pi_c + \tfrac{1}{2}\delta^2\Pi_c. \tag{52}$$

The first variation of Π_c reads:

$$\delta\Pi_c = \int_V \delta\boldsymbol{s} : \boldsymbol{D}^{-1} : \mathscr{R}(\boldsymbol{s}^*)\,\mathrm{d}V - \int_{S_u} \bar{\boldsymbol{u}} \cdot \mathscr{R}(\delta\boldsymbol{s}) \cdot \boldsymbol{n}\,\mathrm{d}S + \int_V \boldsymbol{u}^* \cdot \mathrm{div}\,\mathscr{R}(\delta\boldsymbol{s})\,\mathrm{d}V - \int_{S_t} \boldsymbol{u}^* \cdot \mathscr{R}(\delta\boldsymbol{s}) \cdot \boldsymbol{n}\,\mathrm{d}S, \tag{53}$$

where the third and fourth integrals on the right-hand side are identically vanishing, but have been appended for convenience. In fact, noting that $\boldsymbol{u}^*$ is continuously differentiable, using the Green identity (5) for the first integral on the right-hand side of Eq. (53) and applying the divergence theorem, one has:

$$\delta\Pi_c = \int_V (\boldsymbol{D}^{-1} : \boldsymbol{s}^* - \nabla^s\boldsymbol{u}^*) : \mathscr{R}(\delta\boldsymbol{s})\,\mathrm{d}V + \int_{S_u} (\boldsymbol{u}^* - \bar{\boldsymbol{u}}) \cdot \mathscr{R}(\delta\boldsymbol{s}) \cdot \boldsymbol{n}\,\mathrm{d}S \tag{54}$$

and thus $\delta\Pi_c = 0$ by Eqs. (50), (39c) and (39d). Additionally:

$$\delta^2\Pi_c = \int_V \int_V A(\boldsymbol{x},\boldsymbol{x}')\delta\boldsymbol{s}(\boldsymbol{x}) : \boldsymbol{D}^{-1} : \delta\boldsymbol{s}(\boldsymbol{x}')\,\mathrm{d}V'\,\mathrm{d}V > 0 \tag{55}$$

for any nontrivial $\delta\boldsymbol{s}$. Therefore, Eq. (52) yields:

$$\Pi_c[\boldsymbol{s}] = \Pi_c[\boldsymbol{s}^*] + \tfrac{1}{2}\delta^2\Pi_c \geqslant \Pi_c[\boldsymbol{s}^*] \quad \forall \boldsymbol{s} \in \mathscr{H}, \tag{56}$$

where the equality sign holds if and only if, $\boldsymbol{s} = \boldsymbol{s}^*$ everywhere in V.

Proof of the second part of the theorem. Let $\boldsymbol{s}^*$ be the/a solution (assumed to exist) of the minimum problem

$$\min_{(\boldsymbol{s})} \Pi_c[\boldsymbol{s}] \quad \text{s.t. } \boldsymbol{s} \in \mathscr{H}. \tag{57}$$

In order to transform this constrained problem into an unconstrained one, the Lagrange multiplier method is applied. Then, the relevant Lagrangian functional is written as

$$\Pi_c^L = \Pi_c[\boldsymbol{s}] + \int_V \boldsymbol{u}^* \cdot [\mathrm{div}\,\mathscr{R}(\boldsymbol{s}) + \bar{\boldsymbol{b}}]\mathrm{d}V - \int_{S_t} \boldsymbol{u}^* \cdot [\mathscr{R}(\boldsymbol{s}) \cdot \boldsymbol{n} - \bar{\boldsymbol{t}}]\mathrm{d}S, \tag{58}$$

where the Lagrange multiplier, $\boldsymbol{u}^*$, is specified in $V \cup S_t$ and is assumed C^1-continuous in V. Then, the first variation of Eq. (58) reads, with some mathematics (including application of the divergence theorem):

$$\begin{aligned}\delta\Pi_c^L = &\int_V \mathscr{R}(\delta\boldsymbol{s}) : \left[\boldsymbol{D}^{-1} : \boldsymbol{s}^* - \nabla^s\boldsymbol{u}^*\right]\mathrm{d}V + \int_{S_u} \boldsymbol{n}\cdot\mathscr{R}(\delta\boldsymbol{s})\cdot(\boldsymbol{u}^* - \bar{\boldsymbol{u}})\,\mathrm{d}S + \int_V \delta\boldsymbol{u}^*\cdot[\mathrm{div}\,\mathscr{R}(\boldsymbol{s}^*) + \bar{\boldsymbol{b}}]\mathrm{d}V \\ &- \int_{S_t} \delta\boldsymbol{u}^*\cdot[\boldsymbol{n}\cdot\mathscr{R}(\boldsymbol{s}^*) - \bar{\boldsymbol{t}}]\mathrm{d}S.\end{aligned} \tag{59}$$

As $\delta\Pi_c^L$ must vanish for arbitrary choices of $\delta\boldsymbol{s}$ in $V \cup \partial V$, from Eq. (58) one can state that, in the given hypothesis, there exists a displacement field $\boldsymbol{u}^*$ specified in $V \cup \partial V$ (i.e. the Lagrange multiplier) such that the set $(\boldsymbol{u}^*, \boldsymbol{\varepsilon}^* = \nabla^s\boldsymbol{u}^*, \boldsymbol{\sigma}^* = \mathscr{R}(\boldsymbol{s}^*))$ solves the nonlocal elasticity problem (39a)–(39e), (and problem (57) has unique solution). The proof is so completed. □

5.3. The Hu–Washizu principle for nonlocal elasticity

The functional to consider is

$$\Pi_H := \frac{1}{2}\int_V\int_V A(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{\varepsilon}(\boldsymbol{x}) : \boldsymbol{D} : \boldsymbol{\varepsilon}(\boldsymbol{x}')\,\mathrm{d}V'\,\mathrm{d}V - \int_V[\boldsymbol{\sigma} : (\boldsymbol{\varepsilon} - \nabla^s\boldsymbol{u}) + \bar{\boldsymbol{b}}\cdot\boldsymbol{u}]\mathrm{d}V - \int_{S_u}\boldsymbol{t}\cdot(\boldsymbol{u} - \bar{\boldsymbol{u}})\,\mathrm{d}S - \int_{S_t}\bar{\boldsymbol{t}}\cdot\boldsymbol{u}\,\mathrm{d}S \tag{60}$$

where $\boldsymbol{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$, $\boldsymbol{t}$ are independent fields.

The following can be proved:

Theorem 3. *The solution* $(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ *to the nonlocal elasticity problem* (39a)–(39e), *together with* $\boldsymbol{t} = \boldsymbol{\sigma}\cdot\boldsymbol{n}$ *on* ∂V, *makes stationary the functional* (60); *conversely, the stationarity solution to Eq.* (60) *solves the nonlocal elasticity problem.*

Proof. The first variation of Eq. (60) can be written in the following form:

$$\begin{aligned}\delta\Pi_H = &\int_V \delta\boldsymbol{\varepsilon}(\boldsymbol{x}) : \left[\int_V A(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{D} : \boldsymbol{\varepsilon}(\boldsymbol{x}')\,\mathrm{d}V - \boldsymbol{\sigma}(\boldsymbol{x})\right]\mathrm{d}V - \int_V \delta\boldsymbol{\sigma} : [\boldsymbol{\varepsilon} - \nabla^s\boldsymbol{u}]\mathrm{d}V - \int_V \delta\boldsymbol{u}\cdot[\mathrm{div}\,\boldsymbol{\sigma} + \bar{\boldsymbol{b}}]\mathrm{d}V \\ &+ \int_{S_t}\delta\boldsymbol{u}\cdot[\boldsymbol{\sigma}\cdot\boldsymbol{n} - \bar{\boldsymbol{t}}]\mathrm{d}S + \int_{S_u}\delta\boldsymbol{u}\cdot[\boldsymbol{\sigma}\cdot\boldsymbol{n} - \boldsymbol{t}]\mathrm{d}S - \int_{S_u}\delta\boldsymbol{t}\cdot[\boldsymbol{u} - \bar{\boldsymbol{u}}]\mathrm{d}S.\end{aligned} \tag{61}$$

If the set ($\boldsymbol{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$) solves problem (39a)–(39e), it results that $\delta\Pi_H = 0$ for arbitrary variations $\delta\boldsymbol{u}$, $\delta\boldsymbol{\varepsilon}$, $\delta\boldsymbol{\sigma}$, $\delta\boldsymbol{t}$; that is, the functional Π_H is correspondingly stationary. On the other hand, if the set ($\boldsymbol{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$, $\boldsymbol{t}$) makes stationary Π_H, then $\delta\Pi_H$ must vanish identically, that is for arbitrary choices of the variation fields, which implies that the square-bracketed expressions on the right-hand side of Eq. (61) must each vanish in the respective integration domain. The equations so obtained, the relevant Euler–Lagrange equations, are easily recognized to coincide with Eqs. (39a)–(39e), which therefore are satisfied by the considered functions set. The theorem is so proven. □

Other types of variational principles can be envisaged, but this point is not further investigated here. The extension to nonlocal hyperelastic material is almost straightforward (see Polizzotto (2001) for the Hu–Washizu principle).

6. Nonlocal-FEM-based solution method

The nonlocal elasticity problem of Section 4, Eqs. (39a)–(39e), can be solved numerically by a nonlocal-type finite element method (NL-FEM) technique. Making use of compatible finite elements (FEs), the total potential energy principle of Section 5.1 is employed to establish a related variationally consistent discretization procedure. To this purpose, let the domain V be divided into FEs with subdomains V_n $(n = 1, 2, \dots, N_e)$, and let the unknown displacement field, $\boldsymbol{u}(\boldsymbol{x})$, be represented as

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{N}_n(\boldsymbol{x})\boldsymbol{d}_n \quad \forall \boldsymbol{x} \in V_n, \tag{62}$$

where $\boldsymbol{d}_n$ is a column matrix collecting the node displacements of the nth FE and $\boldsymbol{N}_n(\boldsymbol{x})$ is a suitable matrix of C^0-continuous shape functions with first partial derivatives in L_2. All the arrays $\boldsymbol{d}_n$ are referred to the same Cartesian axes x_i. Thus, the strain field $\boldsymbol{\varepsilon} = \nabla^{\mathrm{s}}\boldsymbol{u}$ has the representation

$$\boldsymbol{\varepsilon}(\boldsymbol{x}) = \boldsymbol{B}_n(\boldsymbol{x})\boldsymbol{d}_n \quad \forall \boldsymbol{x} \in V_n, \tag{63}$$

where $\boldsymbol{B}_n(\boldsymbol{x}) := \nabla^{\mathrm{s}}\boldsymbol{N}_n(\boldsymbol{x})$.

Substituting Eqs. (62) and (63) into Eq. (42) gives

$$\begin{aligned} \Pi^h = {} & \sum_{n=1}^{N_e}\sum_{m=1}^{N_e}\frac{1}{2}\boldsymbol{d}_n^{\mathrm{T}}\left(\int_{V_n}\int_{V_m} A(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{B}_n^{\mathrm{T}}(\boldsymbol{x}) : \boldsymbol{D} : \boldsymbol{B}_m(\boldsymbol{x}')\,\mathrm{d}V'\,\mathrm{d}V\right)\boldsymbol{d}_m \\ & - \sum_{n=1}^{N_e}\boldsymbol{d}_n^{\mathrm{T}}\left(\int_{V_n}\boldsymbol{N}_n^{\mathrm{T}}\cdot\bar{\boldsymbol{b}}\,\mathrm{d}V + \int_{S_{t(n)}}\boldsymbol{N}_n^{\mathrm{T}}\cdot\bar{\boldsymbol{t}}\,\mathrm{d}S\right), \end{aligned} \tag{64}$$

where $S_{t(n)} := S_t \cap \partial V_n$. Denoting by $\boldsymbol{U}$ the column matrix of the node displacements of the whole mesh, one can write:

$$\boldsymbol{d}_n = \boldsymbol{C}_n\boldsymbol{U}, \quad (n = 1, 2, \dots, N_e), \tag{65}$$

where the matrices $\boldsymbol{C}_n$ are the node connection matrices. Thus, substituting Eq. (65) in Eq. (64) one obtains:

$$\Pi^h = \hat{\Pi}(\boldsymbol{U}) \equiv \tfrac{1}{2}\boldsymbol{U}^{\mathrm{T}}\hat{\boldsymbol{K}}\boldsymbol{U} - \boldsymbol{F}^{\mathrm{T}}\boldsymbol{U}, \tag{66}$$

where the following positions hold:

$$\hat{\boldsymbol{K}} := \sum_{n=1}^{N_e}\sum_{m=1}^{N_e}\boldsymbol{C}_n^{\mathrm{T}}\left(\int_{V_n}\int_{V_m} A(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{B}_n^{\mathrm{T}}(\boldsymbol{x}) : \boldsymbol{D} : \boldsymbol{B}_m(\boldsymbol{x}')\,\mathrm{d}V'\,\mathrm{d}V\right)\boldsymbol{C}_m, \tag{67a}$$

$$\boldsymbol{F} := \sum_{n=1}^{N_e}\boldsymbol{C}_n^{\mathrm{T}}\left(\int_{V_n}\boldsymbol{N}_n^{\mathrm{T}}(\boldsymbol{x})\cdot\bar{\boldsymbol{b}}(\boldsymbol{x})\,\mathrm{d}V + \int_{S_{t(n)}}\boldsymbol{N}_n^{\mathrm{T}}(\boldsymbol{x})\cdot\bar{\boldsymbol{t}}(\boldsymbol{x})\,\mathrm{d}S\right), \tag{67b}$$

in which $\hat{\boldsymbol{K}}$ denotes the *nonlocal* stiffness matrix of the mesh and $\boldsymbol{F}$ is the nodal load vector. Eq. (67b) shows that $\boldsymbol{F}$ is the same as with the standard FEM and that the regularity restrictions to be imposed on $\bar{\boldsymbol{b}}$ and $\bar{\boldsymbol{t}}$ within the continuum approach can be relaxed with the present NL-FEM technique.

The solving equation system can be obtained from Eq. (66) by minimizing the quadratic function $\hat{\Pi}(\boldsymbol{U})$, hence

$$\hat{\boldsymbol{K}}\boldsymbol{U} = \boldsymbol{F}. \tag{68}$$

This linear equation system is in all similar to one obtained with the standard FEM, except for $\hat{\boldsymbol{K}}$ which reflects all the nonlocality characteristics of the problem (the superposed hat aims to signal it). $\hat{\boldsymbol{K}}$ is symmetric, positive semi-definite and is in general the result of contributions from all FEs, as in fact Eq. (67a) can be written

$$\hat{\boldsymbol{K}} = \sum_{n=1}^{N_e} \sum_{m=1}^{N_e} \hat{\boldsymbol{K}}_{nm}, \tag{69}$$

where

$$\hat{\boldsymbol{K}}_{nm} := \boldsymbol{C}_n^{\mathrm{T}} \hat{\boldsymbol{k}}_{nm} \boldsymbol{C}_m \tag{70}$$

and

$$\hat{\boldsymbol{k}}_{nm} := \int_{V_n} \int_{V_m} A(\boldsymbol{x}, \boldsymbol{x}') \boldsymbol{B}_n^{\mathrm{T}}(\boldsymbol{x}) : \boldsymbol{D} : \boldsymbol{B}_m(\boldsymbol{x}') \, \mathrm{d}V' \, \mathrm{d}V. \tag{71}$$

Eqs. (69)–(71) show that the element stiffness matrix $\hat{\boldsymbol{k}}_{nm}$ (with dimensions as the number of degrees of freedom of the individual elements) are each enlarged to $\hat{\boldsymbol{K}}_{nm}$ having dimensions as the overall system, Eq. (70), and then assembled altogether to generate $\hat{\boldsymbol{K}}$, Eq. (69). The set $\hat{\boldsymbol{K}}_{nm}$ (or $\hat{\boldsymbol{k}}_{nm}$), $(m = 1, 2, \ldots, N_e)$, includes the element stiffness matrices of the nth FE; in particular: $\hat{\boldsymbol{K}}_{nn}$ is the 'direct'- or 'self'-stiffness matrix of FE n, whereas $\hat{\boldsymbol{K}}_{nm} = \hat{\boldsymbol{K}}_{mn}$ is the 'indirect'- or 'cross'-stiffness matrix of FEs n and m. The cross-stiffness matrix $\hat{\boldsymbol{K}}_{nm}$ is in practice vanishing when the related FEs are too far from each other with respect to the influence distance, R, such that the matrix $\hat{\boldsymbol{K}}$ turns out to be banded in practice, with a band width larger than in the standard FEM (which admits only local-type self-stiffness matrices). A hypothetical nonlocal FEs library will likely include only self-stiffness matrices, the cross-stiffness matrices being strictly related to the mesh geometry.

This topic is being further investigated and developed for a computer program implementation, but is not further pursued here.

7. Standard-FEM-based solution method

In this section, the boundary-value problem of Section 4 is suitably modified in order to make applicable the standard FEM.

7.1. Modified boundary-value problem

The nonlocal elasticity problem (39a)–(39e) is rewritten in an alternative equivalent form centered upon the following *local elasticity* problem with an imposed-like strain, $\boldsymbol{\varepsilon}^{\mathrm{c}}$, i.e.

$$\operatorname{div} \boldsymbol{\sigma} + \bar{\boldsymbol{b}} = 0 \quad \text{in } V \ \text{(field equilibrium)}, \tag{72a}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{t}} \quad \text{on } S_t \ \text{(boundary equilibrium)}, \tag{72b}$$

$$\boldsymbol{\varepsilon} = \nabla^{\mathrm{s}} \boldsymbol{u} \quad \text{in } V \ \text{(field compatibility)}, \tag{72c}$$

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \quad \text{on } S_u \ \text{(boundary compatibility)}, \tag{72d}$$

$$\boldsymbol{\sigma} = \boldsymbol{D} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{c}}) \quad \text{in } V \ \text{(local elasticity)}. \tag{72e}$$

Here, Eqs. (72a)–(72d) are the same as (39a)–(39d), but are here rewritten for ease. (Note that, for $\boldsymbol{\varepsilon}^c$ fixed, the latter equation set can be easily solved, for instance by the standard FEM.) The role of $\boldsymbol{\varepsilon}^c$, here called (*nonlocality*) *correction strain*, is that of simulating the real nonlocality effects in a fictitious local continuum obeying the Hooke law (72e). This goal is achieved by appending to Eqs. (72a)–(72e) the following *consistency* equations, i.e.

$$\boldsymbol{e} = \boldsymbol{\varepsilon} \quad \text{in } V \tag{73}$$

$$\boldsymbol{\varepsilon}^c = \boldsymbol{e} - \mathscr{R}(\boldsymbol{e}) \quad \text{in } V, \tag{74}$$

where $\boldsymbol{e}$ is a (free) *auxiliary strain*. The equation set (72a)–(72e) to (74), obviously equivalent to Eqs. (39a)–(39e), admits variational principles which can be obtained from those of Section 5 through suitable transformations. Here, only the total potential energy principle of Section 5.1 is considered.

To this purpose, let one consider, in place of Eq. (42), the functional

$$\Pi_{\text{md}}[\boldsymbol{u}, \boldsymbol{e}, \boldsymbol{\varepsilon}^c] = \Pi[\boldsymbol{u}, \boldsymbol{\varepsilon}^c] + \Pi_0[\boldsymbol{e}, \boldsymbol{\varepsilon}^c], \tag{75}$$

where

$$\Pi[\boldsymbol{u}, \boldsymbol{\varepsilon}^c] := \frac{1}{2}\int_V \nabla^s\boldsymbol{u} : \boldsymbol{D} : \nabla^s\boldsymbol{u}\,\mathrm{d}V - \int_V \boldsymbol{\varepsilon}^c : \boldsymbol{D} : \nabla^s\boldsymbol{u}\,\mathrm{d}V - \int_V \bar{\boldsymbol{b}} \cdot \boldsymbol{u}\,\mathrm{d}V - \int_{S_t} \bar{\boldsymbol{t}} \cdot \boldsymbol{u}\,\mathrm{d}S, \tag{76a}$$

$$\Pi_0[\boldsymbol{e}, \boldsymbol{\varepsilon}^c] := \frac{1}{2}\int_V \int_V A(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{e}(\boldsymbol{x}) : \boldsymbol{D} : \boldsymbol{e}(\boldsymbol{x}')\,\mathrm{d}V'\,\mathrm{d}V - \frac{1}{2}\int_V \boldsymbol{e} : \boldsymbol{D} : \boldsymbol{e}\,\mathrm{d}V + \int_V \boldsymbol{\varepsilon}^c : \boldsymbol{D} : \boldsymbol{e}\,\mathrm{d}V. \tag{76b}$$

The functional $\Pi[\boldsymbol{u}, \boldsymbol{\varepsilon}^c]$ in Eq. (76a) is recognized as the total potential energy of the body considered made of local elastic material and incorporating an imposed-like strain $\boldsymbol{\varepsilon}^c$; $\Pi_0[\boldsymbol{e}, \boldsymbol{\varepsilon}^c]$ in Eq. (76b) represents a mixed nonlocal/local strain energy related to the auxiliary strain $\boldsymbol{e}$, and the correction strain, $\boldsymbol{\varepsilon}^c$. In the above, $\boldsymbol{u}$ belongs to the class $\mathscr{K}$ defined in Section 5.1 (i.e. $\boldsymbol{u}$ is C^1-continuous and satisfies Eq. (72d)), whereas $\boldsymbol{e}$ and $\boldsymbol{\varepsilon}^c$ belong to the class, $\mathscr{C}_0$, of C^0-continuous second order symmetric tensors; or, in abriged form, the set $\boldsymbol{y} := (\boldsymbol{u}, \boldsymbol{e}, \boldsymbol{\varepsilon}^c)$ belongs to the class $\mathscr{Y} = \mathscr{K} \times \mathscr{C}_0^2$, i.e.

$$\mathscr{Y} = \{\boldsymbol{y} = (\boldsymbol{u}, \boldsymbol{e}, \boldsymbol{\varepsilon}^c) : \boldsymbol{u} \in \mathscr{K},\ \boldsymbol{e} \in \mathscr{C}_0, \boldsymbol{\varepsilon}^c \in \mathscr{C}_0\}. \tag{77}$$

The first variation of Eq. (75), taking into account Eqs. (76a) and (76b) and after some mathematics including the divergence theorem, can be written as follows:

$$\begin{aligned}\delta\Pi_{\text{md}} = {} & -\int_V \delta\boldsymbol{u} \cdot \{\mathrm{div}[\boldsymbol{D} : (\nabla^s\boldsymbol{u} - \boldsymbol{\varepsilon}^c)] + \bar{\boldsymbol{b}}\}\,\mathrm{d}V + \int_{\partial V} \delta\boldsymbol{u} \cdot \{\boldsymbol{n} \cdot \boldsymbol{D} : (\nabla^s\boldsymbol{u} - \boldsymbol{\varepsilon}^c) - \bar{\boldsymbol{t}}\}\,\mathrm{d}S \\ & - \int_V \delta\boldsymbol{e} : \boldsymbol{D} : \{\boldsymbol{e} - \mathscr{R}(\boldsymbol{e}) - \boldsymbol{\varepsilon}^c\}\,\mathrm{d}V + \int_V \delta\boldsymbol{\varepsilon}^c : \boldsymbol{D} : \{\boldsymbol{e} - \nabla^s\boldsymbol{u}\}\,\mathrm{d}V.\end{aligned} \tag{78}$$

At the stationarity condition for $\Pi_{\text{md}}[\boldsymbol{y}]$ in $\mathscr{Y}$, $\delta\Pi_{\text{md}}$ must vanish for arbitrary choices of the variation fields, with $\delta\boldsymbol{u} \in \mathscr{K}'$ (i.e. $\delta\boldsymbol{u} = \boldsymbol{0}$ on S_u). It results that all the braced expressions of Eq. (78) must vanish in the relevant integration domains and that thus the stationarity solution $(\boldsymbol{y}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$, where $\boldsymbol{y} = (\boldsymbol{u}, \boldsymbol{e}, \boldsymbol{\varepsilon}^c)$, $\boldsymbol{\varepsilon} := \nabla^s\boldsymbol{u}$, $\boldsymbol{\sigma} := \boldsymbol{D} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^c)$, if exists, solves also the equation set (72a)–(72e) to (74). The converse is also valid, as in fact, if the set $(\boldsymbol{y}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$, with $\boldsymbol{y} \in \mathscr{Y}$ and $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$ in the class $\mathscr{C}_0$, satisfies Eqs. (72a)–(72e) to (74), it is $\delta\Pi_{\text{md}} = 0$ identically for arbitrary variation fields (but $\delta\boldsymbol{u} = \boldsymbol{0}$ on S_u) and thus $\Pi_{\text{md}}[\boldsymbol{y}]$ is stationary in $\mathscr{Y}$.

As a consequence of the above, it is possible to state the following:

Total potential energy principle for local elasticity with nonlocality correction strain. If there exists a set $\boldsymbol{y} = (\boldsymbol{u}, \boldsymbol{e}, \boldsymbol{\varepsilon}^c) \in \mathscr{Y}$ that makes $\Pi_{\text{md}}[\boldsymbol{y}]$ stationary in $\mathscr{Y}$, then the set $(\boldsymbol{y}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ with $\boldsymbol{\varepsilon} := \nabla^s\boldsymbol{u}$ and $\boldsymbol{\sigma} := \boldsymbol{D} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^c)$, solves problem (72a)–(72e) to (74) and hence also problem (39a)–(39e). Conversely, if the set $(\boldsymbol{y}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$, with $\boldsymbol{y} = (\boldsymbol{u}, \boldsymbol{e}, \boldsymbol{\varepsilon}^c) \in \mathscr{Y}$ and $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$ in $\mathscr{C}_0$, is a solution of problem (72a)–(72e) to (74), hence also of problem (39a)–(39e), then $\boldsymbol{y}$ makes $\Pi_{\text{md}}[\boldsymbol{y}]$ stationary in $\mathscr{Y}$.

7.2. A solving iterative procedure

One may wish to catch the stationarity conditions for $\Pi_{\text{md}}[\boldsymbol{y}]$, and thus the solution to the nonlocal boundary-value problem (39a)–(39e), through an iterative procedure in which the fields $\boldsymbol{e}$, $\boldsymbol{\varepsilon}^c$ are taken fixed at every iteration. Then, because $\delta\boldsymbol{e} = \delta\boldsymbol{\varepsilon}^c = 0$ identically within every iteration, the first variation $\delta\Pi_{\text{md}}$ in Eq. (78) loses the last two integrals and thus the stationarity conditions reduce to Eqs. (72a)–(72e) with $\boldsymbol{\varepsilon}^c$ at the fixed value. The latter equation set, being a problem of local elasticity, can be easily solved by a standard FEM technique. Then the computed solution can be used with the consistency equations (73) and (74) to derive new (hopefully improved) fields $\boldsymbol{e}$, $\boldsymbol{\varepsilon}^c$ to be employed in a new iteration. In this way, an iterative procedure of the type local prediction/nonlocal correction is generated.

Let $k = 1, 2, \ldots$ be such an iterative sequence and let $\boldsymbol{\varepsilon}^c = \boldsymbol{\varepsilon}^c_{(k-1)}$ be an approximate value of the correction strain (likely obtained in the course of the $(k-1)$th iteration). With $\boldsymbol{\varepsilon}^c$ taken fixed at this value, let problem (72a)–(72e) be solved to obtain the displacement field $\boldsymbol{u}_{(k)}$ and the related strain field, $\boldsymbol{\varepsilon}_{(k)}$. Applying the standard FEM together with the interpolation formulae of Section 6, the solving equation system can be written as follows:

$$\boldsymbol{K}\boldsymbol{U}_{(k)} = \boldsymbol{F} + \boldsymbol{F}^c_{(k-1)}, \tag{79}$$

where $\boldsymbol{K}$ is the standard (local-type) stiffness matrix, i.e.

$$\boldsymbol{K} = \sum_{n=1}^{N_e} \boldsymbol{C}_n^{\mathrm{T}} \boldsymbol{k}_n \boldsymbol{C}_n, \tag{80a}$$

$$\boldsymbol{k}_n = \int_{V_n} \boldsymbol{B}_n^{\mathrm{T}} : \boldsymbol{D} : \boldsymbol{B}_n \mathrm{d}V \quad (n = 1, 2, \ldots, N_e). \tag{80b}$$

Also, $\boldsymbol{F}$ is the load vector of Eq. (67b), whereas $\boldsymbol{F}^c_{(k-1)}$ is a fictitious force vector (here called *correction force vector*) representative of the correction strain and given by

$$\boldsymbol{F}^c_{(k-1)} = \sum_{n=1}^{N_e} \boldsymbol{C}_n^{\mathrm{T}} \int_{V_n} \boldsymbol{B}_n^{\mathrm{T}} : \boldsymbol{D} : \boldsymbol{\varepsilon}^c_{(k-1)} \, \mathrm{d}V. \tag{81}$$

The solution $\boldsymbol{U}_{(k)}$ to the equation set (79) is the *local prediction* in the kth iteration. The related *nonlocal correction* consists in computing a new value of $\boldsymbol{\varepsilon}^c$, say $\boldsymbol{\varepsilon}^c_{(k)}$, by means of Eqs. (73) and (74). Taking, for instance, $\boldsymbol{e}$ coincident with $\boldsymbol{\varepsilon}_{(k)}$, one can write

$$\boldsymbol{\varepsilon}^c_{(k)} = \boldsymbol{\varepsilon}_{(k)} - \mathscr{R}(\boldsymbol{\varepsilon}_{(k)}) \quad \text{in } V \tag{82}$$

where the field $\boldsymbol{\varepsilon}_{(k)}$ is derived from $\boldsymbol{U}_{(k)}$ by Eqs. (63) and (65); then, an updated correction vector, say $\boldsymbol{F}^c_{(k)}$, can be computed by replacing $\boldsymbol{\varepsilon}^c_{(k-1)}$ of Eq. (81) with $\boldsymbol{\varepsilon}^c_{(k)}$. A new iteration can then be started.

The procedure initiates with $\boldsymbol{\varepsilon}^c_{(0)}$ and $\boldsymbol{F}^c_{(0)}$ vanishing, such that $\boldsymbol{U}_{(1)}$ is the local elasticity solution of the given problem. Note that, whenever – likely in the first few iterations – the compatible field $\boldsymbol{\varepsilon}_{(k)}$ in Eq. (82) exhibits a singularity, it can be regularized for use in the latter equation (due to the fact that $\boldsymbol{e}$ and $\boldsymbol{\varepsilon}$ are independent fields in the above stationarity principle).

No attempt is made here to show the convergence of the proposed iterative procedure, that is, that $\boldsymbol{U}_{(k)}, k = 1, 2, \ldots$ converges to $\boldsymbol{U}$, the solution to Eq. (68). The numerical performance of the proposed method is being studied separately.

8. Comments and conclusion

The intent of this paper was to study the boundary-value problem for nonlocal elasticity in the hypothesis of infinitesimal displacements and quasi-static loads. To this purpose the Eringen (1972a,b, 1978, 1979) model has been refined by introducing the concept of 'geodetical distance' to replace the Euclidean distance on which usually the attenuation function is assumed to depend. The underlying idea is that the nonlocality effects propagate along geodetical paths from the source points to the others, such as not to traverse obstacles (if any) as cracks, holes, and more in general any gap in the convexity of the domain. The refined diffusion law is just a conjecture, though quite reasonable, as no experimental results seem to be available. Such a refinement may play an important role within fracture mechanics, where the use of the geodetical distance inside a region enclosing the crack is expected to lead to improvements. The extent in which this may be true and the practical ways to account for the geodetical distance are issues open to future research.

The L_2-space of square integrable functions has been invoked in order to realize a connection with the theory of integral equations, which offers some theorems capable to guarantee a well-behaved constitutive model (one-to-one correspondence between local strain and nonlocal stress fields, and positive-definite strain energy). This, in turn, implies that the nonlocal elasticity theory be imbedded in the L_2 function space, which imposes some restrictions to the load data fields, besides the unknown fields.

The refined model has been framed within thermodynamics with nonlocality features, characterized by: (i) the lack of the local actions principle, by which the first principle of thermodynamics, written in pointwise form, contains an additional term (the nonlocality energy residual) accounting for the energy interchanges between the material particles due to the diffusion processes of the nonlocality effects; (ii) the second principle (entropy production rate inequality) holding in pointwise form to correctly assess the irreversibility/reversibility properties of the material constitutive behavior. In this way, not only the constitutive equations of nonlocal elasticity have been obtained as the pertinent state equations, but also the energy residual has been evaluated and interpreted as for its thermodynamic role.

The nonlocal elasticity boundary-value problem has been shown to admit a unique solution and three variational principles have been presented to characterize it. These principles are extensions to nonlocal elasticity of classical principles of local elasticity, that is, the total potential energy, the complementary energy and the mixed Hu–Washizu principles. The former principle has been used to derive a nonlocal FEM, referred to as NL-FEM in this paper. The Hu–Washizu principle has been used to derive a (symmetric) boundary element method for nonlocal elasticity elsewhere (Polizzotto, 2001).

The lack of regularity often exhibited by load data (body and surface forces) usually adopted in practice may pose serious difficulties as for the continuum solution existence. A typical example is the impossibility to construct a Kelvin-type fundamental solution for nonlocal elasticity (Eringen model) due to the impossibility for a (quite regular) nonlocal stress field to equilibrate a point concentrated load. This undesirable feature of nonlocal elasticity can perhaps be overcome by a suitable definition of the loading data fields, for instance considering the body and surface forces as the nonlocal counterparts of some local data (the latter being not restricted in regularity); but this point is not elaborated here.

The NL-FEM envisaged in this paper is based on compatible FEs. The related (linear) solving equation system is formally the same as for the standard FEM, but the relevant global (nonlocal) symmetric stiffness matrix reflects the nonlocality features of the problem. This nonlocal stiffness matrix is the result of contributions from all FEs, each of which contributes with a single self-stiffness matrix and a set of as many cross-stiffness matrices as there are other FEs in the mesh. As in practice the cross-stiffness matrix of two FEs located too far from each other (with respect to the influence distance) is vanishing, the global matrix is banded, the band width being in general larger than for the standard FEM. This NL-FEM appears to be notably more complex than the standard FEM, but nevertheless it may result an effective analysis tool against the complexity of analytical solution methods even of simple continuum problems (Eringen, 1972a, 1978; Eringen et al., 1977).

The standard FEM has been shown to be also applicable to solve the nonlocal elasticity problem. This is rendered possible by an iterative procedure of the type local prediction/nonlocal correction in which the nonlocality is simulated by an imposed-like correction strain. In every iteration, a local problem is solved by a standard FEM technique and for a fixed correction strain; then the latter is updated making use of the computed local solution. This procedure, which possesses a firm variational basis, is expected to converge to the same solution obtained by the NL-FEM.

The main purpose of this paper was to contribute to the development of the continuum mechanics aspects of nonlocal elasticity. This goal has been reached by establishing some theoretical results and giving some basic ideas for the numerical solution methods of the relevant boundary-value problem. These results are being further elaborated and improved in an ongoing research work.

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Appendix A

A compact notation is used, with bold-face letters for vectors and tensors. The 'dot' and 'colon' products between vectors and tensors denote the simple and double index contraction operations, respectively, e.g. $\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i$, $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ji}$, $\boldsymbol{n}\cdot\boldsymbol{\sigma} = \{n_j\sigma_{ji}\}$, $\boldsymbol{D} : \boldsymbol{\varepsilon} = \{D_{ijkh}\varepsilon_{hk}\}$, where the subscripts denote Cartesian components and the repeated index summation rule is applied. Cartesian orthogonal co-ordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ are employed. Partial derivatives with respect to x_j are denoted by commas, e.g. $u_{i,j} = \partial u_i/\partial x_j$. The symbol $:=$ means equality by definition. Other symbols will be defined in the text where they appear for the first time.

References

Altan, S.B., 1989a. Uniqueness of the initial-value problems in nonlocal elastic solids. Int. J. Solids Struct. 25, 1271–1278.

Altan, S.B., 1989b. Existence in nonlocal elasticity. Arch. Mech. 41, 25–36.

Altan, S.B., 1991. Uniqueness in nonlocal thermoelasticity. J. Therm. Stresses 14, 121–128.

Artan, R., Yelkenci, T., 1996. Rectangular rigid stamp on nonlocal elastic half plane. Int. J. Solids Struct. 33, 3577–3586.

Borino, G., Fuschi, P., Polizzotto, C., 1999. A thermodynamic approach to nonlocal plasticity and related variational principles. J. Appl. Mech. ASME 66, 952–963.

Coleman, B.D., Gurtin, M., 1967. Thermodynamics with internal variables. J. Chem. Phys. 47, 597–613.

Edelen, D.G.B., Laws, N., 1971. On the thermodynamics of systems with nonlocality. Arch. Rat. Mech. Anal. 43, 24–35.

Edelen, D.G.B., Green, A.E., Laws, N., 1971. Nonlocal continuum mechanics. Arch. Rat. Mech. Anal. 43, 36–44.

Eringen, A.C., 1972a. Nonlocal polar elastic continua. Int. J. Engng. Sci. 10, 1–16.

Eringen, A.C., 1972b. Linear theory of nonlocal elasticity and dispersion of plane waves. Int. J. Engng. Sci. 10, 425–435.

Eringen, A.C., 1976. Nonlocal micropolar field theory. In: Eringen, A.C. (Ed.), Continuum Physics, vol. 4. Acadamic Press, New York, pp. 205–267.

Eringen, A.C., 1978. Line crack subjected to shear. Int. J. Fract. 14, 367–379.

Eringen, A.C., 1979. Line crack subjected to anti-plane shear. Engng. Fract. Mech. 12, 211–219.
Eringen, A.C., 1981. On nonlocal plasticity. Int. J. Engng. Sci. 12, 1461–1474.
Eringen, A.C., 1983. Theories of nonlocal plasticity. Int. J. Engng. Sci. 21, 741–751.
Eringen, A.C., 1987. Theory of nonlocal elasticity and some applications. Res. Mechanica 21, 313–342.
Eringen, A.C., Edelen, D.G.B., 1972. On nonlocal elasticity. Int. J. Engng. Sci. 10, 233–248.
Eringen, A.C., Kim, B.S., 1974. Stress concentration at the tip of a crack. Mech. Res. Commun. 1, 233–237.
Eringen, A.C., Speziale, C.G., Kim, B.S., 1977. Crack-tip problem in non-local elasticity. J. Mech. Phys. Solids 25, 339–355.
Gao, J., 1999. An asymmetric theory of nonlocal elasticity – part 2: continuum field. Int. J. Solids Struct. 36, 2959–2971.
Germain, P., Nguyen, Q.S., Suquet, P., 1983. Continuum thermodynamics. J. Appl. Mech. ASME 50, 1010–1021.
Green, A.E., Rivlin, R.S., 1965. Multipolar continuum mechanics: functional theory I. Proc. Royal Soc. A 284, 303–324.
Gutkin, M.Yu., Aifantis, E.C., 1996. Screw dislocation in gradient elasticity. Scripta Materialia 35, 1353–1358.
Kröner, E., 1967. Elasticity theory of materials with long range cohesive forces. Int. J. Solids Struct. 3, 731–742.
Krumhansl, J.A., 1968. Some considerations on the relations between solid state physics and generalized continuum mechanics. In: Kröner, E. (Ed.), Mechanics of Generalized Continua. Springer, Berlin, pp. 298–331.
Lemaitre, J., Chaboche, J.-L., 1990. Mechanics of Solid Materials. Cambridge University Press, Cambridge.
Polizzotto, C., Borino, G., 1998. Thermodynamics-based formulation of gradient plasticity. Eur. J. Mech. A/Solids 17, 741–761.
Polizzotto, C., Borino, G., Fuschi, P., 2000. Shakedown of cracked bodies with nonlocal elasticity. CD-Rom Proceedings of ECCOMAS-2000, CIMNE, Barcelona.
Polizzotto, C., 2001. BEM techniques in nonlocal elasticity. In: Hall, W.S., Oliveto, G. (Eds.), Boundary Element Methods for Soil-Structure Interaction, Kluwer Academic Publishers, Dordrecht, The Netherlands, in press.
Rogula, D., 1982. Introduction to nonlocal theory of material media. In: Rogula, D. (Ed.), Nonlocal Theory of Material Media. Springer, Berlin, pp. 125–222.
Tricomi, F.G., 1985. Integral Equations. Dover Publications, New York.
Vardulakis, I., Exadaktylos, G., Aifantis, E.C., 1996. Gradient elasticity with surface energy: mode III crack problem. Int. J. Solids Struct. 33, 4531–4559.
Wang, J., Dhaliwal, R.S., 1993a. On some theorems in nonlocal theory of micropolar elasticity. Int. J. Solids Struct. 30, 1331–1338.
Wang, J., Dhaliwal, R.S., 1993b. Uniqueness in generalized nonlocal thermoelasticity. J. Therm. Stresses 16, 71–78.
Washizu, K., 1982. Variational Methods in Elasticity and Plasticity, third ed. Pergamon Press, New York.
Zhou, Z.-G., Han, J.-C., Du, S.-Y., 1999. Investigation of a Griffith crack subjected to anti-plane shear by using the non-local theory. Int. J. Solids Struct. 36, 3891–3901.